#### Lecture 10

Mathematical Induction





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$$P(3) \land \dots \land P(n) \land \dots$$

$$P(3) \land \dots \land P(n-1) \to P(n), \dots$$

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**Note:** Mathematical Induction can also be applied to prove statements on other domains, mathematical objects such as Graphs, or prove correctness of algorithms, etc.

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**Basis Step:** For n = 8,  $3^8 = 6561 > 8^4 = 4096$ .

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 $3k^4 > (k+1)^4 \quad \Longleftrightarrow \quad \left(\frac{k}{k+1}\right)^4 > \frac{1}{3}$ 



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$$\left(1 - \frac{1}{k+1}\right)^4 = (8/9)^4$$





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 $.624 > \frac{1}{3}$ 





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Suppose the tournament had 4 players and the following are the results of all matches.

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